*Int. J. Solids Structures*, 1972, Vol. 8, pp. 699 to 707. Pergamon Press. Printed in Great Britain

# ELASTIC WAVES PRODUCED BY LONGITUDINAL IMPACT ON A SYSTEM WITH SYMMETRICALLY BRANCHED RODS

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**Abstract**—The problem of the propagation of a longitudinal elastic wave which is travelling along a thin uniform rod at the end of which a branched, symmetrically arranged system of rods is joined is treated analytically. The symmetry condition greatly simplifies the analysis and enables the problem to be treated as a two-dimensional one. The theory used for the propagation of longitudinal waves along the rods is the one-dimensional one while flexural wave propagation is treated using the Timoshenko theory. The numerical results obtained with this analysis are found to agree well with experimental observations.

## NOTATION

| | |
|---|---|
| $A_0, A_r, A_T, B_j, H$ | arbitrary functions of angular frequency |
| $c, c' = \sqrt{(E'/\rho')}$ | bar velocity, $E'$ is Young's modulus and $\rho'$ is the density |
| $c_1, c_2$ | flexural wave propagation velocity of first and second mode according to the Timoshenko theory |
| $k = \sqrt{(I/\Omega)}$ | radius of gyration, $I$ moment of inertia of the cross section, $\Omega$ cross sectional area |
| $k'G'$ | effective shear modulus in the branches |
| $n$ | number of branches |
| $P$ | angular frequency |
| $T$ | time |
| $u$ | longitudinal displacement |
| $u_0, u_r, u_T$ | longitudinal displacement of incident, reflected and transmitted waves |
| $v$ | lateral displacement |
| $v_b$ | lateral displacement due to bending |
| $v_s$ | lateral displacement due to shear |
| $x, y, \zeta$ | coordinate systems |
| $\alpha$ | nondimensional ratio |
| $\rho, \rho'$ | density of main rod and its branches respectively |
| $\varepsilon'$ | nondimensional parameter depends on the Poisson's ratio $\nu$ and shape of the cross-section |

## INTRODUCTION

THE longitudinal elastic impact of a rod with discontinuity in cross-sectional area, elastic modulus and density was treated by Rayleigh in 1894 [1]. He found that such discontinuities in general gave rise to reflected waves. Ripperger and Abramson [2] analyzed the reflection and transmission of flexural waves for the same problem and obtained the coefficients for an infinitely long wave train. In these two cases the axes of the rods were taken as collinear and only waves of the same type were generated. Recently wave propagation in rods of different geometries have been considered. Morley [3] solved the problem of a naturally curved rod and derived a Timoshenko-type equation for rods with large and small curvatures. Lee and Kolsky [4] solved the problem of a sharply bent rod subjected to longitudinal impact. In the former case axial and lateral displacements are coupled everywhere in the rod, while in the latter problem, coupling occurs only at the bend. The purpose of this paper is to investigate the longitudinal impact on a rod symmetrically split into $n$ (integer equal to or greater than 2) branches.

This problem has practical significance; dynamic impact on a beam ($n = 2$) such as the case of a beam supporting a hoist, an elevator car, the simplified problem of firing a rocket symmetrically supported by a tripod ($n = 3$). All serve as good examples of this problem.

## ANALYSIS

In the mathematical treatment that follows it is assumed that a longitudinal elastic wave travels down the rod and is reflected and transmitted at the boundary where the branches separate. The boundary conditions used at the junction are continuity of the displacements and continuity of the stresses.

For the longitudinal waves, a one-dimensional bar theory is used, i.e. it is assumed that

$$\frac{\partial^2 u}{\partial t^2} = c_0^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u$ is the longitudinal displacement at a coordinate $x$ and a time $t$, $c_0 = (E/\rho)^{\frac{1}{2}}$ where $E$ is Young's modulus and $\rho$ is the density of the bar. For the propagation of flexural waves, the Timoshenko equations in coupled form, i.e.

$$\frac{\partial v_s}{\partial x} + k^2 \varepsilon' \frac{\partial^3 v_b}{\partial x^3} - \frac{\varepsilon' k^2}{c_0^2} \frac{\partial^3 v_b}{\partial x \partial t^2} = 0 \tag{2a}$$

$$\frac{\partial^2 v_s}{\partial x^2} - \frac{\varepsilon'}{c_0^2} \frac{\partial^2 v_b}{\partial t^2} - \frac{\varepsilon'}{c_0^2} \frac{\partial^2 v_s}{\partial t^2} = 0 \tag{2b}$$

are used to describe the motion of the rods. In these equations the total deflection $v$ is broken into two parts, one due to bending $v_b$ and the other due to shearing $v_s$, and $v = v_b + v_s$ (see e.g. Miklowitz [5]).

Let us consider an incident sinusoidal wave train of angular frequency $p$ and amplitude $A_0$ propagating in increasing $x$ direction, we have the expression

$$u_0 = A_0 \exp[ip(t - x/c_0)]. \tag{3}$$

As a result of symmetry, the axis of the main rod is restricted from moving laterally and the reflected flexural wave is identically zero. Only a longitudinal wave is reflected back in the main rod, i.e.

$$u_r = A_r \exp[ip(t + x/c_0)]. \tag{4}$$

In each branch, in general, both longitudinal and flexural waves are generated, i.e.

$$u_T = A_T \exp[ip(t - \xi/c_0')] \tag{5}$$

$$\begin{aligned} v &= v_b + v_s = \sum_{j=1}^{2} (1 + \alpha_j) B_j \exp[ip(t - \xi/c_j)] \\ \alpha_j &= \frac{\varepsilon'}{c_0'^2 [c_j^2 - \varepsilon'/c_0'^2]} \qquad (j = 1, 2) \end{aligned} \tag{6}$$

where $c_1$ and $c_2$ are here the phase velocities of the first and the second modes according to the Timoshenko equations (2) and

$$v_b = \sum_{j=1}^{2} B_j \exp[ip(t-\xi/c_j)] \tag{7a}$$

and

$$v_s = \sum_{j=1}^{2} \alpha_j B_j \exp[ip(t-\xi/c_j)] \tag{7b}$$

$A_0, A_r, A_T, B_j$ are functions of $p$ and are complex in nature. $A_0$ has to be determined from the initial condition of impact while $A_r, A_T, B_j$ are determined by the following conditions at $x = \xi = 0$.

(a) displacement in $x$ direction must be continuous, i.e.

$$u_0 - u_r = u_T \cos\theta - v \sin\theta \tag{8}$$

(b) as a result of the symmetry condition, there is no lateral motion of the main rod, hence the resultant displacement of each branch in the $y$ direction must vanish, i.e.

$$u_T \sin\theta + v\cos\theta = 0 \tag{9}$$

(c) since (for a rigid joint) at the boundary the angle of rotation must be zero

$$v_{b,\zeta} = 0 \tag{10}$$

and finally

(d) the forces in the $x$ direction must be balanced, i.e.

$$\rho c\Omega(u_{0,t}+u_{r,t}) = n\rho' c'\Omega' u_{T,t}\cos\theta + nk'G'\Omega' v_{s,\xi}\sin\theta \tag{11}$$

where a comma denotes differentiation with respect to the index which follows.

Due to the symmetry of the system, the moment balance condition and the condition that the forces must be balanced in the $y$ direction are automatically satisfied.

Substituting equations (3)–(7) into equations (8)–(11), we get

$$A_0 - A_r - A_T\cos\theta + \sum_{j=1}^{2}(1+\alpha_j)B_j \sin\theta = 0 \tag{12}$$

$$A_T \sin\theta + \sum_{j=1}^{2}(1+\alpha_j)B_j\cos\theta = 0 \tag{13}$$

$$\sum_{j=1}^{2}\frac{B_j}{c_j} = 0 \tag{14}$$

$$A_0 + A_r - \frac{n\rho' c'\Omega'}{\rho c\Omega}\cos\theta\, A_T + \sum_{j=1}^{2}\frac{nk'G'\Omega'}{\rho c\Omega}\left(\frac{\alpha_j\sin\theta}{c_j}\right)B_j = 0 \tag{15}$$

solving equations (12)–(15) for $A_r$, $A_T$ and $B_j$, we obtain

$$A_r = \left[1 - \frac{2}{1+k_1\cos^2\theta + k_2\sin^2\theta}\right]A_0 \tag{16}$$

$$A_T = \left[\frac{2\cos\theta}{1+k_1\cos^2\theta+k_2\sin^2\theta}\right]A_0 \tag{17}$$

$$B_1 = \frac{2\sin\theta}{k_3(1+k_1\cos^2\theta)+k_4\sin^2\theta}A_0 \tag{18}$$

$$B_2 = \frac{2\sin\theta(c_2/c_1)}{k_3(1+k_1\cos^2\theta)+k_4\sin^2\theta}A_0 \tag{19}$$

where

$$k_1 = \frac{n\rho'c'\Omega'}{\rho c\Omega} \tag{20a}$$

$$k_2 = \frac{nk'G'\Omega'}{\rho c\Omega}\frac{(\alpha_2-\alpha_1)}{(1+\alpha_2)c_2-(1+\alpha_1)c_1} \tag{20b}$$

$$k_3 = \frac{(1+\alpha_2)c_2-(1+\alpha_1)c_1}{c_1} \tag{20c}$$

$$k_4 = \frac{nk'G'\Omega'}{\rho c\Omega}\frac{\alpha_2-\alpha_1}{c_2}. \tag{20d}$$

As may be seen from equations (16)–(19) when $\Omega' = 0$ then $A_r = -A_0$. This result checks with the wave reflected from the free end of a rod. For $\theta = \theta$ we have $B_1 \equiv B_2 \equiv 0$

$$A_r = \frac{n\rho'c'\Omega'-\rho c\Omega}{n\rho'c'\Omega'+\rho c\Omega} \tag{21a}$$

and

$$A_T = \frac{2\rho c\Omega}{n\rho'c'\Omega'+\rho c\Omega} \tag{21b}$$

these results agree with those in [2].

## NUMERICAL SOLUTIONS AND COMPARISON WITH EXPERIMENT

To serve as an example, numerical solutions are obtained in this section and part of the results are compared with Ranganath's experimental results [6]. Ranganath solved the problem of the normal impact of an infinite elastic beam by a semi-infinite elastic rod. In his work, a 10 in. long $\frac{3}{8}$ in. diameter striker rod was accelerated by a gas gun and impinged on a $\frac{3}{8}$ in. transmitter bar of 2 ft length resting on a $\frac{1}{8}\times\frac{3}{8}$ in. long rectangular beam. The incident and reflected longitudinal pulses in the transmitted bar and the flexural pulses transmitted into the beam were then recorded using strain gages. Some of his results are shown in Figs. 4 and 5 for purposes of comparison. Dimensions of the beam and the rod and the Poisson's ratio of the material used were as follows:

thickness of the beam = $\frac{1}{8}$ in.;
width of the beam = $\frac{3}{8}$ in.;
diameter of the transmitted rod = $\frac{3}{8}$ in.;
Poisson's ratio = 0·3;
length of the transmitter rod = 24 in.

The beam is assumed to be very long so that the reflected waves from the ends are not included in the analysis.

To check the validity of the theory presented here, numerical results were obtained using the data given above, so that Ranganath's experimental results could be used for comparison. Instead of using a step load as was assumed in his analysis, an incident pulse with a 8 $\mu$sec rise time and a 60 $\mu$sec duration was used as being closer to reality.

The normalized incident longitudinal pulse can be represented by

$$F(t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \{\exp[-(t-t_1)^2/4a^2] - \exp[-(t-t_2)^2/4a^2]\}\, dt \tag{22}$$

where $a = 10^{-6}$ sec.

To simplify the computation, the pulse was placed at the middle of the chosen fundamental interval $T_0$ (= 600 $\mu$sec), thus all the sine terms vanish and $t_1 = 270\,\mu$sec and $t_2 = 330\,\mu$sec.

The fundamental frequency used was then $1{\cdot}05 \times 10^4$ rad./sec and the summation was taken over the first 200 terms (up to a frequency of $2{\cdot}10 \times 10^6$ rad./sec) which was more than sufficient. It was found that even the summation of the first 100 terms gave fairly accurate results. The cut-off frequency for the second Timoshenko mode is $6{\cdot}27 \times 10^6$ rad./sec, so in this region $c_1$ is real and $c_2 = iv_2$ (where $v_2$ is real) is purely imaginary.

If we let $A_r = A_r^* + iA_r^{**}$, $A_T = A_T^* + iA_T^{**}$, $B_1 = B_1^* + iB_1^{**}$ and $B_2 = B_2^* + iB_2^{**}$, then the normalized reflected longitudinal strain pulse in the main rod and the transmitted longitudinal and flexural strain pulses in each branch are given, respectively, by

$$\varepsilon_R = \sum_{k=1}^{200} \frac{kp_0}{c_0}[A_r^* \sin(kp_0 t) + A_r^{**} \cos(kp_0 t)]A_0^{**}$$

$$\varepsilon_T = \sum_{k=1}^{200} \frac{kp_0}{c}[A_T^* \sin(kp_0 t) + A_T^{**} \cos(kp_0 t)]A_0^{**}$$

and

$$\varepsilon_F = \sum_{k=1}^{200} \frac{k^2 p_0^2}{8}\left\{\frac{1}{c_1^2}[B_1^* \cos(kp_0(t-\xi/c_1)) - B_1^{**} \sin(kp_0(t-\xi/c_2))]\right.$$
$$\left. -(B_2^* \cos kp_0 t - B_2^{**} \sin kp_0 t)\exp(-p\xi/v_2)/v_2^2\right\} A_0^{**}$$

and the results are plotted in Figs. 1–5.

Figure 1 represents the normalized incident and reflected longitudinal strain–time profile for $n = 2$. The strain due to the reflected longitudinal pulse $\varepsilon_r$ is normalized with respect to that of the incident pulse $\varepsilon_I$. It is seen that the symmetrical incident pulse generates non-symmetrical reflected strain–time profiles for all the three cases, i.e. $\theta = \pi/6$, $\pi/3$ and $\pi/2$. In all cases the pulses have rounded corners near the front and have relatively sharp corners in the regions of unloading. Furthermore, in all cases the durations of the pulses are elongated to about 90 $\mu$sec. Figure 2 shows the normalized transmitted longitudinal strain–time profiles for $n = 2$. The pulse shapes take essentially the same forms as those shown in the reflected longitudinal pulses, but in the case of reflected longitudinal pulses the amplitudes increase with the branch angle $\theta$ while the amplitudes of the transmitted longitudinal pulses decrease with increasing angles (for $\theta = \pi/2$ the transmitted longi-

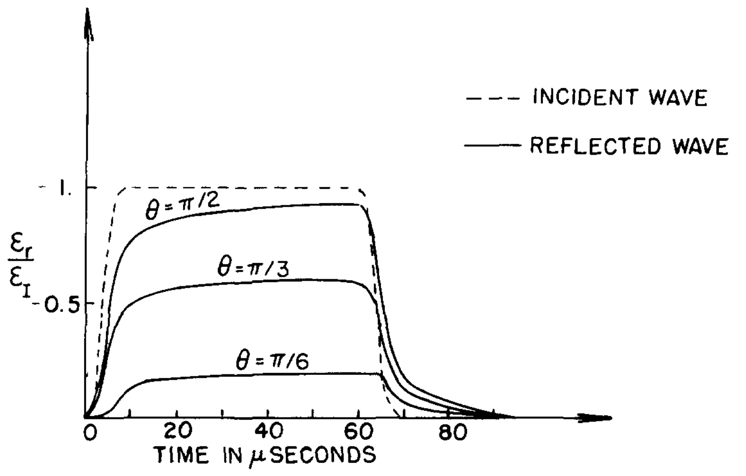


FIG. 1. Incident and reflected longitudinal strain pulses ($n = 2$).

tudinal pulse is identically zero). For the reflected pulses the normalized amplitudes are always less than 1, while for the transmitted pulse the ratio of the amplitudes can be greater than, equal to or less than 1. Which of these occurs depends on the ratios of the cross-sectional areas of the branches to that of the main rod. Figure 3 shows the normalized transmitted flexural pulses at a distance of 1 in. from the branch point. For $\theta = 0$ the flexural pulse is identically zero and the amplitudes of the pulses at other angles increase with increasing angles while the pulse shape at the three angles shown is practically unchanged.

It is interesting to note that for the case $n = 2$ and $\theta = \pi/2$, the problem becomes one of the normal impact of a rod on a beam as presented in [6]. The numerical results obtained from the theory given here are compared with the experimental results as shown in Figs. 4 and 5.

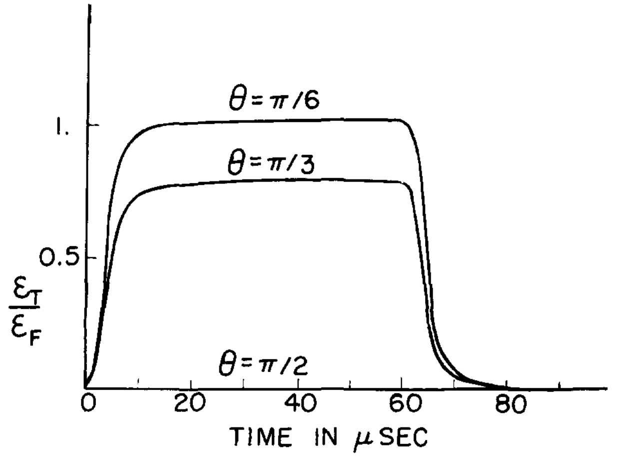


FIG. 2. Transmitted longitudinal strain pulses ($n = 2$).

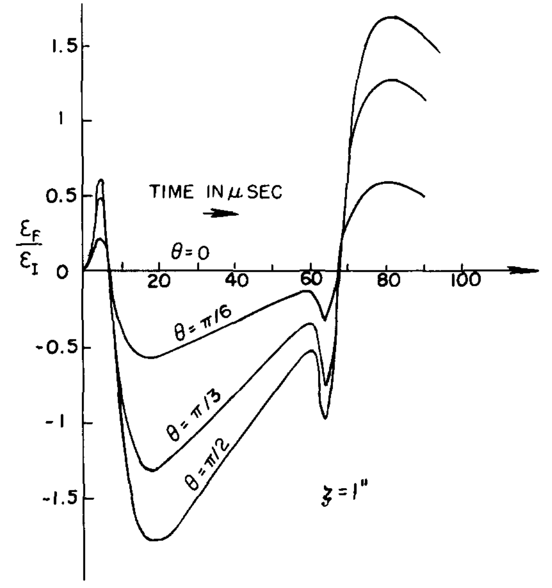


FIG. 3. Transmitted flexural pulses ($n = 2$).

Figure 4 shows a comparison between the experimental results as found in [6] and the theoretical results obtained here and shown in Fig. 1 ($\theta = \pi/2$). The kink in the curve which was observed experimentally is not predicted by this theory, but it was not predicted in the theory given in [6] either. One possible reason for this deviation is the use of the Timoshenko theory which gives only two modes of propagation instead of the infinite number predicted by the exact theory, also the second mode deviates somewhat from the second mode of the exact theory. Some other reasons were discussed in [6].

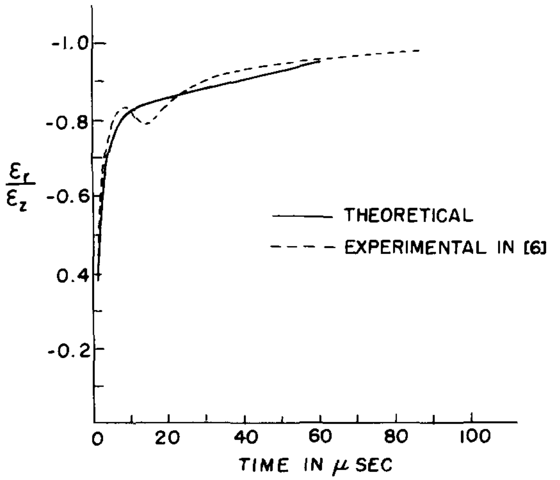


FIG. 4. Comparison of reflected longitudinal pulses.

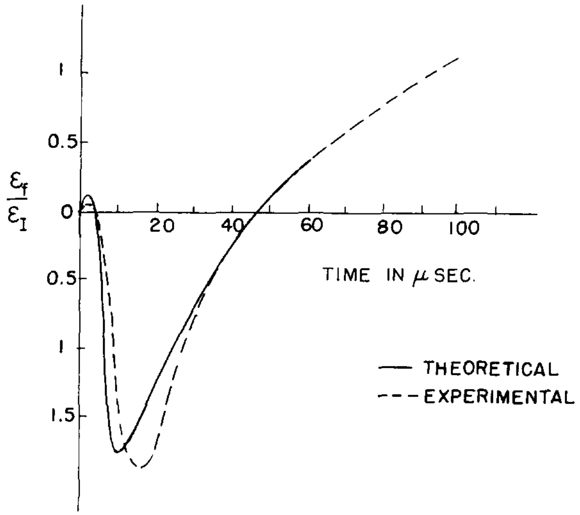


FIG. 5. Comparison of theoretical and experimental strain–time profiles at $\xi = \frac{3}{4}$ in.

## DISCUSSION

The theory presented in this paper predicts the reflected and the transmitted waves generated in a symmetrically branched rod, when a longitudinal wave travels down the rod. By taking advantage of the symmetry, the problem which is in fact three-dimensional can be reduced to a two-dimensional one and the analysis is thus greatly simplified. The analysis is applicable to a rod split into $n$ branches lying symmetrically with respect to the axis of the rod. The branches are assumed to make angles which can vary from 0 to 90° with the axis. For $\theta = 0°$ the results agree with those presented by Ripperger and Abramson [2] and for $\theta = 90°$ the results agree well with the experimental results presented in [6], where the analysis was carried out with a transform technique. It may prove difficult to use such a transform technique when the load function is not a step function. Thus the Fourier synthesis method presented here has certain advantages over the transform method, as it can be used to solve the problem for arbitrary incident pulse shapes.

*Acknowledgements*—The author is greatly indebted to Professor H. Kolsky and Professor D. Y. Hsieh for their kindness in reading an early draft and making some very helpful suggestions. Appreciation is also extended to Mrs. E. Fonseca for careful preparation of the manuscript and to Miss E. Addison for making the drawings. The support of the Army Research Office (Durham) during the course of this research is also gratefully acknowledged.

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## APPENDIX

$$A_r^* = \left[\frac{2\phi}{\phi^2+\psi^2}\right]A_0^{**}$$

$$A_r^{**} = \left[1-\frac{2\psi}{\phi^2+\psi^2}\right]A_0^{**}$$

$$A_T^* = \left[-\frac{2\cos\theta\phi}{\phi^2+\psi^2}\right]A_0^{**}$$

$$A_T^{**} = \left[\frac{2\cos\theta\psi}{\phi^2+\psi^2}\right]A_0^{**}$$

$$B_1^* = \left[\frac{2\sin\theta r}{r^2+\beta^2}\right]A_0^{**}$$

$$B_1^{**} = \left[\frac{2\sin\theta\beta}{r^2+\beta^2}\right]A_0^{**}$$

$$B_2^* = B_1^{**}\frac{v_2}{c_1}$$

and

$$B_2^{**} = -\frac{v_2}{c_1}B_1^*$$

where

$$\phi = -\frac{nk'G'\Omega'}{\rho c\Omega}\frac{(\alpha_1-\alpha_2)(1+\alpha_2)v_2\sin^2\theta}{(1+\alpha_2)^2v_2^2+(1+\alpha_1)^2c_1^2}$$

$$\psi = 1+\frac{n\rho'c'\Omega'}{\rho c\Omega}\cos^2\theta+\frac{nk'G'\Omega'}{\rho c\Omega}\frac{(\alpha_1-\alpha_2)(1+\alpha_1)c_1\sin^2\theta}{(1+\alpha_1)^2c_1^2+(1+\alpha_2)^2v_2^2}$$

$$r = \left(1+\frac{n\rho'c'\Omega'}{\rho c\Omega}\cos^2\theta\right)\frac{(1+\alpha_2)v_2}{c_1}$$

$$\beta = \frac{nk'G'\Omega'}{\rho c\Omega}\frac{(\alpha_2-\alpha_1)\sin^2\theta}{c_1}-\left(1+\frac{n\rho'c'\Omega'}{\rho c\Omega}\cos^2\theta\right)(1+\alpha_1)$$

$$A_0^{**} = \frac{60}{(k\pi)^2}\exp\left[-\left(\frac{k\pi}{300}\right)^2\right]\left(\sin\frac{11\,k\pi}{10}-\sin\frac{9\,k\pi}{10}\right)$$

for the $k$th harmonic.

(*Received* 17 *May* 1971; *revised* 19 *October* 1971)

**Абстракт**—Обсуждается аналитически задача распространения продольной упругой волны, движущейся вдоль тонкого, однородного стержня, на конце которого присоединена разветленная, симметрически расположенная стержневая система. Условие симметрии очень упрощает анализ и дает возможность обработки задачи как двухмерной. Используемая теория распространения продольных волн является одномерной, пока рассматривается распространение волны изгиба на основе теории Тимошенки. Оказывается, что численные результаты, получены по этому анализу, хорошо согласовуются с экспериментальными наблюдениями.